Bogoliubov Theory of Disordered Bose-Einstein Condensates

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We describe interacting bosons at low temperature in spatially correlated random potentials. By a Bogoliubov expansion around the deformed mean-field condensate, the fundamental Hamiltonian for elementary excitations is derived, achieving an analytical formulation in the case of weak disorder. From this, we calculate the sound velocity and true zero-temperature condensate depletion in correlated disorder and all dimensions.

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The interplay between interaction, quantum statistics, and randomness defines one of the richest problems in condensed matter physics: the dirty boson problem [1, 2]. Determining the ground state of a disordered Bose gas is already a formidable task [3]; all the more desirable is an analytical theory for the excitations of disordered Bose-Einstein condensates (BECs). In presence of a well-developed condensate, i.e. at low temperature and for weak disorder, the most economic description, due to Bogoliubov [4], treats quantum fluctuations around the best mean-field approximation to the condensate. One of the key quantities that such a theory should provide is the experimentally measurable, *disorder-renormalized sound velocity*, which characterizes the low-energy dispersion and enters all thermodynamic properties. In this respect, existing theories for bosons in disorder are not entirely satisfactory. While simple approaches cannot determine a change in quasiparticle dispersion at all [5–7], more elaborate calculations find a positive correction to the sound velocity due to uncorrelated disorder [8–10]. Still others, in different settings and with different methods, report a negative correction [11–14]. Here, a unifying framework is missing. On a more conceptual level, the *condensate deformation* by disorder is often confounded with the *condensate depletion*, i.e. the fraction of particles not in the condensate at all. This quantum depletion is a crucial quantity whose smallness validates Bogoliubov's approach; it contains all particles with non-zero momentum [15] *only if* the condensate is homogeneous. If the condensate is deformed, it has non-zero momentum components on its own, already on the mean-field level [16]. Only these have been counted by previous approaches [5–10]. For the true disorder-induced quantum depletion there are, to our knowledge, only few numerical results [17]. An analytical calculation of this pivotal quantity is lacking so far.

With this Letter, we present a comprehensive Bogoliubov theory for inhomogeneous Bose-Einstein condensates that gives access to a wealth of relevant quantities, including the full excitation dispersion with the renormalized speed of sound and the localization length of elementary excitations [18, 19]. Notably, our formulation encompasses the class of spatially correlated disorder that is currently under study with ultracold gases [20–22]. By expanding the many-body Hamiltonian around the deformed mean-field solution, we derive the fundamental Bogoliubov Hamiltonian for excitations, verified to be orthogonal to the ground state. For weak disorder, a fully analytical description is achieved. From this Hamiltonian, we calculate corrections to the sound velocity and zero-temperature quantum depletion for correlated disorder in all dimensions.

STARTING LINE

Interacting bosons are described by

$$\hat{E} = \int \mathrm{d}^d r\, \hat{\Psi}^\dagger(\boldsymbol{r}) \left[h(\boldsymbol{r}) + \frac{g}{2} \hat{\Psi}(\boldsymbol{r})^\dagger \hat{\Psi}(\boldsymbol{r}) \right] \hat{\Psi}(\boldsymbol{r}), \qquad (1)$$

with the grand-canonical single-particle Hamiltonian

$$h(\boldsymbol{r}) = -\frac{\hbar^2}{2m}\nabla^2 + V(\boldsymbol{r}) - \mu \qquad (2)$$

and field operators that obey $\left[\hat{\Psi}(\boldsymbol{r}), \hat{\Psi}^\dagger(\boldsymbol{r}')\right] = \delta(\boldsymbol{r} - \boldsymbol{r}')$ [15]. The global confining potential is assumed to be very smooth, ideally a very large box, and $V(\boldsymbol{r})$ describes local spatial fluctuations. Repulsive interaction between bosons is accounted for by $g = 4\pi\hbar^2 a_\mathrm{s}/m > 0$ (in $d = 3$, with s-wave scattering length a_s), an excellent approximation for cold and dilute gases, where the gas parameter $(n a_\mathrm{s}^3)^{1/2}$ is small.

Below a critical temperature, the Bose gas forms a BEC, i.e. a large fraction of particles condense into the ground state of the single-particle density matrix. In the absence of interaction, this is the ground state of the potential $V(\boldsymbol{r})$. Also interacting bosons condense, into a mode whose shape results from the competition between kinetic energy, confinement and interaction. Bogoliubov theory [4] takes advantage of this macroscopic occupation and splits the quantum field into a mean-field condensate and quantized fluctuations: $\hat{\Psi}(\boldsymbol{r}) = \Phi(\boldsymbol{r}) + \delta\hat{\Psi}(\boldsymbol{r})$.

CONDENSATE

We first describe how a weak external potential deforms the condensate. By definition, the ground state minimizes the energy (1) on the mean-field level and thus obeys the stationary Gross-Pitaevskii (GP) equation $h(\boldsymbol{r})\Phi(\boldsymbol{r})+g\Phi(\boldsymbol{r})^3=0$ [15]. The condensate's kinetic energy is minimized by choosing a fixed global phase, and we can take $\Phi(\boldsymbol{r})$ real. The imprint of a weak potential on the condensate amplitude can be computed perturbatively by expanding $\Phi(\boldsymbol{r})=\sqrt{n}+\Phi^{(1)}(\boldsymbol{r})+\Phi^{(2)}(\boldsymbol{r})+\dots$ in powers of V around the homogeneous solution $\Phi^{(0)}=\sqrt{n}$ [11, 16]. In order to maintain a fixed average density n, also the chemical potential is adjusted at each order, $\mu=gn+\mu^{(1)}+\mu^{(2)}+\dots$. We insert these expansions into the GP equation, transform to momentum representation, and collect orders up to V^2. The first-order imprint then is

$$\Phi^{(1)}_{\boldsymbol{q}}=-\frac{(1-\delta_{\boldsymbol{q}0})V_{\boldsymbol{q}}}{2gn+\epsilon^0_q}N^{1/2}. \tag{3}$$

This linear-response deformation is proportional to the potential's matrix element $V_{\boldsymbol{q}}=L^{-d}\int \mathrm{d}^d r e^{-i\boldsymbol{q}\cdot\boldsymbol{r}}V(\boldsymbol{r})$. In the denominator of eq. (3), the bare kinetic energy $\epsilon^0_q=\hbar^2q^2/2m$ equals the interaction energy gn when q^{-1} equals the BEC healing length $\xi=\hbar/\sqrt{2mgn}$. Thus, the condensate readily follows potential components with $q\xi\ll 1$, but shows a strongly smoothed imprint when $q\xi\gg 1$ [16]. Pushing the expansion to second order, we find

$$\Phi^{(2)}_{\boldsymbol{q}}=\frac{1}{N^{1/2}}\sum_{\boldsymbol{p}}\Phi^{(1)}_{\boldsymbol{q}-\boldsymbol{p}}\Phi^{(1)}_{\boldsymbol{p}}\frac{(1-\delta_{\boldsymbol{q}0})\epsilon^0_p-gn}{2gn+\epsilon^0_q}. \tag{4}$$

Eqs. (3) and (4) determine the disorder imprints also on derived quantities like the density $n_{\boldsymbol{k}}=L^{-d}\sum_{\boldsymbol{q}}\Phi_{\boldsymbol{k}-\boldsymbol{q}}\Phi_{\boldsymbol{q}}$.

FLUCTUATIONS

We expand the Hamiltonian (1) in powers of $\delta\hat\Psi$ and $\delta\hat\Psi^\dagger$ around the condensate. The linear term vanishes, because $\Phi(\boldsymbol{r})$ minimizes the energy functional. The relevant contribution is then the quadratic part that can be readily expressed in density-phase variables $\delta\hat\Psi=\delta\hat n(\boldsymbol{r})/2\Phi(\boldsymbol{r})+i\Phi(\boldsymbol{r})\delta\hat\varphi(\boldsymbol{r})$ [23]:

$$\hat H=\int \mathrm{d}^d r\Big\{\frac{\hbar^2}{2m}\Big[\Big(\boldsymbol{\nabla}\frac{\delta\hat n}{2\Phi(\boldsymbol{r})}\Big)^2+\frac{\big[\nabla^2\Phi(\boldsymbol{r})\big]}{4\Phi^3(\boldsymbol{r})}\delta\hat n^2 + \Phi^2(\boldsymbol{r})(\boldsymbol{\nabla}\delta\hat\varphi)^2\Big]+\frac{g}{2}\delta\hat n^2\Big\}. \tag{5}$$

In a homogeneous system with $\Phi(\boldsymbol{r})=\sqrt{n}$, it is advisable to transform to Fourier space and Bogoliubov excitations

$$\hat\gamma_{\boldsymbol{k}}=\frac{\delta\hat n_{\boldsymbol{k}}}{2a_k\sqrt{n}}+ia_k\sqrt{n}\,\delta\hat\varphi_{\boldsymbol{k}}. \tag{6}$$

These bosonic excitations obey $[\hat\gamma_{\boldsymbol{k}},\hat\gamma^\dagger_{\boldsymbol{k}'}]=\delta_{\boldsymbol{k}\boldsymbol{k}'}$. Moreover, by choosing $a_k=(\epsilon^0_k/\epsilon_k)^{1/2}$ the homogeneous Hamiltonian becomes diagonal,

$$\hat H^{(0)}=\sum_{\boldsymbol{k}}\epsilon_k\hat\gamma^\dagger_{\boldsymbol{k}}\hat\gamma_{\boldsymbol{k}}, \tag{7}$$

with $\epsilon_k=[\epsilon^0_k(2gn+\epsilon^0_k)]^{1/2}$ the clean Bogoliubov dispersion relation [4]. Characteristically, low-energy excitations ($k\xi\ll 1$) have the linear dispersion $\epsilon_k=\hbar ck$, with sound velocity $c=\sqrt{gn/m}$.

In presence of disorder, two things change. First of all, the reference point, from where the excitations originate, is shifted to the inhomogeneous ground state $\Phi(\boldsymbol{r})$. But still, eq. (6) is fit to define excitations with the proper commutation relations. The density-phase representation (6) implies also that the fluctuation

$$\delta\hat\Psi(\boldsymbol{r})=\sum_{\boldsymbol{k}}[u_{\boldsymbol{k}}(\boldsymbol{r})\hat\gamma_{\boldsymbol{k}}-v_{\boldsymbol{k}}(\boldsymbol{r})^*\hat\gamma^\dagger_{\boldsymbol{k}}] \tag{8}$$

decomposes over excitations with modes

$$u_{\boldsymbol{k}}(\boldsymbol{r})=\frac{1}{2L^{d/2}}\left[\frac{\Phi(\boldsymbol{r})}{a_k\sqrt{n}}+\frac{a_k\sqrt{n}}{\Phi(\boldsymbol{r})}\right]e^{i\boldsymbol{k}\cdot\boldsymbol{r}}, \tag{9}$$

$$v_{\boldsymbol{k}}(\boldsymbol{r})=\frac{1}{2L^{d/2}}\left[\frac{\Phi(\boldsymbol{r})}{a_k\sqrt{n}}-\frac{a_k\sqrt{n}}{\Phi(\boldsymbol{r})}\right]e^{i\boldsymbol{k}\cdot\boldsymbol{r}}. \tag{10}$$

For the clean system with $\Phi(\boldsymbol{r})=\sqrt{n}$, these modes reduce to plane waves with the well-known amplitudes $u_k=\frac{1}{2}(a_k^{-1}+a_k)$ and $v_k=\frac{1}{2}(a_k^{-1}-a_k)$ such that $u_k^2-v_k^2=1$. In presence of disorder, the modes (9) and (10) are defined such that they still satisfy the biorthogonality

$$\int \mathrm{d}^d r\,[u^*_{\boldsymbol{k}}(\boldsymbol{r})u_{\boldsymbol{k}'}(\boldsymbol{r})-v^*_{\boldsymbol{k}}(\boldsymbol{r})v_{\boldsymbol{k}'}(\boldsymbol{r})]=\delta_{\boldsymbol{k}\boldsymbol{k}'} \tag{11}$$

required for eigenmodes of the Bogoliubov Hamiltonian. Moreover, the excitations are also orthogonal to the deformed ground state, because $\Phi(\boldsymbol{r})\,[u_{\boldsymbol{k}}(\boldsymbol{r})-v_{\boldsymbol{k}}(\boldsymbol{r})]$ is a plane wave with zero average for all $\boldsymbol{k}\neq 0$ [24, 25]. These are crucial properties for the low-energy excitations of the system to be well defined [26].

As a second difference to the homogeneous case, these excitations now live on a deformed background. Both differences can be accounted for by defining a single effective potential $\mathcal{V}_{\boldsymbol{k}\boldsymbol{k}'}=\left(\begin{smallmatrix}W & Y\\ Y & W\end{smallmatrix}\right)_{\boldsymbol{k}\boldsymbol{k}'}$ that mediates scattering between the different components of the Bogoliubov-Nambu pseudo spinor $\hat\Gamma_{\boldsymbol{k}}=(\hat\gamma_{\boldsymbol{k}},\hat\gamma^\dagger_{-\boldsymbol{k}})^{\mathrm{t}}$. From (5), we thus arrive at the inhomogeneous Bogoliubov Hamiltonian

$$\hat H=\frac{1}{2}\sum_{\boldsymbol{k}}\epsilon_k\hat\Gamma^\dagger_{\boldsymbol{k}}\hat\Gamma_{\boldsymbol{k}}+\frac{1}{2}\sum_{\boldsymbol{k},\boldsymbol{k}'}\hat\Gamma^\dagger_{\boldsymbol{k}}\mathcal{V}_{\boldsymbol{k}\boldsymbol{k}'}\hat\Gamma_{\boldsymbol{k}'} \tag{12}$$

with the structure $\hat H=\hat H^{(0)}+\hat H^{(V)}$. At this point, the only approximation made is the neglect of third and

fourth order terms in the fluctuations. In contrast, $\hat{H}^{(V)}$ is still exact in the disorder strength.

A perturbative, but fully analytical description is obtained by expanding $\mathcal{V} = \left(\begin{smallmatrix} W & Y \\ Y & W \end{smallmatrix}\right) = \mathcal{V}^{(1)} + \mathcal{V}^{(2)} + \dots$ to lowest orders in the bare disorder with the help of eqs. (3) and (4). The small parameter of this expansion is $v = V/gn \ll 1$. The first-order scattering amplitudes $W^{(1)}_{\boldsymbol{k}\boldsymbol{k}'} = w^{(1)}_{\boldsymbol{k}\boldsymbol{k}'} V_{\boldsymbol{k}-\boldsymbol{k}'}$ and $Y^{(1)}_{\boldsymbol{k}\boldsymbol{k}'} = y^{(1)}_{\boldsymbol{k}\boldsymbol{k}'} V_{\boldsymbol{k}-\boldsymbol{k}'}$ are proportional to $V_{\boldsymbol{k}-\boldsymbol{k}'}$, as required by conservation of momentum. All information about interaction and condensate background is factorized into amplitude envelopes

$$\begin{aligned} w^{(1)}_{\boldsymbol{k}\boldsymbol{k}'} &= \frac{a_k a_{k'} \xi^2 (1-\delta_{\boldsymbol{k}\boldsymbol{k}'})}{2+\xi^2(\boldsymbol{k}'-\boldsymbol{k})^2} \left[k^2 + k'^2 - \boldsymbol{k}\cdot\boldsymbol{k}' - \frac{\boldsymbol{k}\cdot\boldsymbol{k}'}{a_k^2 a_{k'}^2} \right], \\ y^{(1)}_{\boldsymbol{k}\boldsymbol{k}'} &= \frac{a_k a_{k'} \xi^2 (1-\delta_{\boldsymbol{k}\boldsymbol{k}'})}{2+\xi^2(\boldsymbol{k}'-\boldsymbol{k})^2} \left[k^2 + k'^2 - \boldsymbol{k}\cdot\boldsymbol{k}' + \frac{\boldsymbol{k}\cdot\boldsymbol{k}'}{a_k^2 a_{k'}^2} \right]. \end{aligned} \tag{13}$$

Second-order scattering amplitudes are later only needed for $\boldsymbol{k} = \boldsymbol{k}'$ because the disorder average restores translation invariance: $W^{(2)}_{\boldsymbol{k}\boldsymbol{k}} = Y^{(2)}_{\boldsymbol{k}\boldsymbol{k}} = \sum_{\boldsymbol{p}} w^{(2)}_{\boldsymbol{k}\boldsymbol{p}} V_{\boldsymbol{k}-\boldsymbol{p}} V_{\boldsymbol{p}-\boldsymbol{k}}$, with

$$w^{(2)}_{\boldsymbol{k}\boldsymbol{p}} = \frac{a_k^2 \xi^2}{2gn} \frac{p^2 + 3(\boldsymbol{k}-\boldsymbol{p})^2 + 3k^2}{[2+(\boldsymbol{k}-\boldsymbol{p})^2\xi^2]^2} (1-\delta_{\boldsymbol{k}\boldsymbol{p}}). \tag{14}$$

This concludes our derivation of the inhomogeneous Bogoliubov Hamiltonian. From here, one can derive numerous physical quantities for any given potential $V(\boldsymbol{r})$. For notational simplicity only, we assume in the following that $V(\boldsymbol{r})$ describes disorder that is homogeneous and isotropic under the ensemble average, with $\overline{V} = 0$ and

$$\overline{V_{\boldsymbol{q}} V_{-\boldsymbol{q}'}} = L^{-d} \delta_{\boldsymbol{q}\boldsymbol{q}'} V^2 \sigma^d C_d(q\sigma). \tag{15}$$

The dimensionless function $C_d(q\sigma)$ characterizes the potential correlations persisting on the length scale σ; the normalization is chosen such that in the thermodynamic limit $\int \frac{\mathrm{d}^d u}{(2\pi)^d} C_d(u) = 1$.

LOCALIZATION LENGTH

The Hamiltonian (12) is a random operator, varying with each realization of the quenched disorder potential. Therefore, the Bogoliubov excitations in $d = 1$ are expected to be localized by the disorder [18, 19]. And indeed, we can calculate their localization length as $l_{\mathrm{loc}}^{-1} = \gamma_{2k}/2v_{\mathrm{g}}$ from the backscattering rate γ_{2k} and group velocity $\hbar v_{\mathrm{g}} = \partial_k \epsilon_k$. To lowest order in the small parameter $v = V/gn \ll 1$, the backscattering rate derived by Fermi's Golden Rule from the Hamiltonian (12) reads

$$\gamma_{2k} = 2\pi\rho(\epsilon_k) \overline{W^{(1)\,2}_{k(-k)}}, \tag{16}$$

where the density of states is $\rho(\epsilon_k) = [\pi\hbar v_{\mathrm{g}}]^{-1}$. The resulting $l_{\mathrm{loc}}^{-1} = \frac{1}{4} v^2 k^2 \sigma C_1(2k\sigma)/(1+k^2\xi^2)^2$ agrees perfectly with [18, 19]. Importantly, it is characteristic for sound waves that localization is less pronounced at low energy. Indeed, for $k\xi \to 0$ at a fixed correlation ratio $\zeta = \sigma/\xi$ of order unity, the localization rate per wave length vanishes like $[kl_{\mathrm{loc}}]^{-1} \sim v^2 k\xi \to 0$. Consequently, the low-energy properties of the interacting quantum gas are not affected by localization. The underlying reason is that the disorder is screened by interaction [11]—contrary to the case of noninteracting particles, where localization is stronger at lower energy [27].

In higher dimensions, localization is even less pronounced, with the localization length being exponentially large compared to the mean free path, if not infinite. Low-energy excitations are free to propagate over long times and large distances. The main effect of disorder then is to renormalize the excitation dispersion relation.

DISORDER-MODIFIED DISPERSION

The disorder-modified quasiparticle dispersion $\bar{\epsilon}_k = \epsilon_k + \Delta\bar{\epsilon}_k$ can be determined by applying standard Nambu-Green perturbation theory [28] to the relevant Hamiltonian (12). Thanks to its perturbative structure $\hat{H} = \hat{H}^{(0)} + \hat{H}^{(V)}$, it is straightforward to calculate the self-energy of the single-excitation Green function [26]. The self-energy's imaginary part provides the elastic scattering rate, which vanishes at low energy just like the localization rate discussed above. From the self-energy's real part, we deduce the shift in the dispersion

$$\frac{\Delta\bar{\epsilon}_k}{\epsilon_k} = v^2 \sigma^d \int \frac{\mathrm{d}^d q}{(2\pi)^d} z_{\boldsymbol{k}\boldsymbol{q}} C_d(q\sigma), \tag{17}$$

with the kernel (P denotes the principal value)

$$z_{\boldsymbol{k}\boldsymbol{q}} = \frac{g^2 n^2}{\epsilon_k} \left[\mathrm{P}\, \frac{[w^{(1)}_{\boldsymbol{k}(\boldsymbol{k}+\boldsymbol{q})}]^2}{\epsilon_k - \epsilon_{\boldsymbol{k}+\boldsymbol{q}}} - \frac{[y^{(1)}_{\boldsymbol{k}(\boldsymbol{k}+\boldsymbol{q})}]^2}{\epsilon_k + \epsilon_{\boldsymbol{k}+\boldsymbol{q}}} + w^{(2)}_{\boldsymbol{k}(\boldsymbol{k}+\boldsymbol{q})} \right]. \tag{18}$$

Together with expressions (13) and (14), eq. (17) allows calculating the dispersion of Bogoliubov excitations in weak, but arbitrarily correlated disorder.

In the hydrodynamic limit $\xi \to 0$, where the healing length is shorter than both correlation length σ and wavelength $2\pi/k$, eq. (18) simplifies considerably, and eq. (17) reproduces eq. (28) of Ref.[14]. In this regime, the excitation energy is *reduced* in all dimensions and for any value of $k\sigma$. For low energies $k\sigma \to 0$ and smooth potentials $\sigma \gg \xi$, the sound-velocity shift $\Delta\bar{c}/c = -v^2/(2d)$ is independent of the correlation details.

We now change the point of view by taking the limit $k \to 0$, with an arbitrary correlation ratio $\zeta = \sigma/\xi$. In particular, this allows us to reach the case of δ-correlated disorder where $\sigma \ll \xi, k^{-1}$. The kernel (18) simplifies to

$$z_{0\boldsymbol{q}} = 2\frac{q^2\xi^2 - (2+q^2\xi^2)\cos^2\beta}{(2+q^2\xi^2)^3}, \quad \beta = \measuredangle(\boldsymbol{k},\boldsymbol{q}). \tag{19}$$

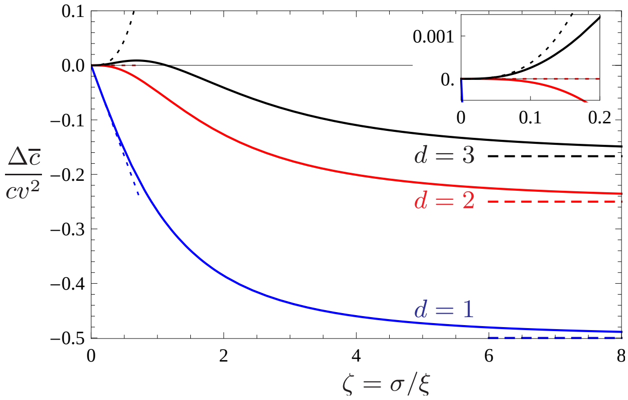

FIG. 1. (Color online) Sound-velocity correction, computed from (17) and (19), due to Gaussian correlated disorder, eq. (20), with variance $v^2 := V^2/(gn)^2 \ll 1$ as function of correlation ratio $\zeta = \sigma/\xi$. For most values, the sound velocity is reduced. Dashed and dotted: universal limits for very smooth ($\zeta \gg 1$) and δ-correlated disorder ($\zeta \ll 1$) respectively, as collected in Tab. I. Inset: same data around the origin, showing the rapid departure from the leading-order estimate in $d = 3$ [8–10].

$\Delta\bar{c}/cv^2$	$d = 1$	$d = 2$	$d = 3$
$\zeta \gg 1$	$-\frac{1}{2}$	$-\frac{1}{4}$	$-\frac{1}{6}$
$\zeta \ll 1$	$-\frac{3C_1(0)\zeta}{16\sqrt{2}}$	0	$+\frac{5C_3(0)\zeta^3}{48\sqrt{2}\pi}$

TABLE I. Universal limits of the speed-of-sound correction, computed from (17) and (19), for very smooth disorder ($\zeta \gg 1$) and δ-correlated disorder ($\zeta \ll 1$). See also footnote .

Fig. 1 shows the disorder correction (17) to the speed of sound, resulting from kernel (19) and a generic Gaussian pair correlation

$$C_d(q\sigma) = (2\pi)^{d/2} \exp\{-q^2\sigma^2/2\}. \tag{20}$$

The plotted curves can be expressed in closed form, but the details depend on the specific correlator and are not of general interest. In contrast, one finds universal behavior for very small or very large ζ. The limit $\zeta \to \infty$ of a very smooth potential coincides, as it should, with the hydrodynamic limit $\Delta\bar{c}/c = -v^2/(2d)$. In the opposite limit $\zeta \to 0$ of δ-correlated disorder, the correlator $C_d(0)$ can be pulled out of the integral (17), which becomes elementary. The correction then scales as ζ^d, as shown in Fig. 1, with prefactors that are collected in Tab. I. [29] Notably, we corroborate the known result for δ-correlated disorder in $d = 3$ [8–10], the only case with a positive correction. But our theory reveals that this estimate is of limited use because already a small correlation makes a large difference, as shown by the inset of Fig. 1.

CONDENSATE DEPLETION

Finally, we investigate the condensate depletion properly speaking, namely the density of particles out of the mean-field condensate, $\delta n := L^{-d} \int \mathrm{d}^d r \langle \delta\hat{\Psi}(\boldsymbol{r})^\dagger \delta\hat{\Psi}(\boldsymbol{r})\rangle$. Inserting (8) and rearranging terms, we find that the depletion density can be written

$$\delta n = \frac{1}{4nL^d} \sum_{\boldsymbol{k},\boldsymbol{k}'} \Big\{ \Big[a_k a_{k'} \check{n}_{\boldsymbol{k}'-\boldsymbol{k}} + \frac{n_{\boldsymbol{k}'-\boldsymbol{k}}}{a_k a_{k'}} \Big] \langle \hat{\gamma}_{\boldsymbol{k}} \hat{\gamma}^\dagger_{\boldsymbol{k}'} + \hat{\gamma}^\dagger_{\boldsymbol{k}'} \hat{\gamma}_{\boldsymbol{k}} \rangle$$
$$+ \Big[a_k a_{k'} \check{n}_{\boldsymbol{k}'-\boldsymbol{k}} - \frac{n_{\boldsymbol{k}'-\boldsymbol{k}}}{a_k a_{k'}} \Big] \langle \hat{\gamma}_{\boldsymbol{k}} \hat{\gamma}_{-\boldsymbol{k}'} + \hat{\gamma}^\dagger_{\boldsymbol{k}'} \hat{\gamma}^\dagger_{-\boldsymbol{k}} \rangle - 2\delta_{\boldsymbol{k}\boldsymbol{k}'} \Big\}. \tag{21}$$

In principle, this expression is correct to all orders in V. One only requires the Fourier components $n_{\boldsymbol{k}} := [\Phi(\boldsymbol{r})^2]_{\boldsymbol{k}} = L^{-d} \sum_{\boldsymbol{q}} \Phi_{\boldsymbol{k}-\boldsymbol{q}} \Phi_{\boldsymbol{q}}$ of the deformed mean-field condensate density, as well as the Fourier components of its inverse, $\check{n}_{\boldsymbol{k}} := n^2[\Phi(\boldsymbol{r})^{-2}]_{\boldsymbol{k}}$. Up to order V^2, they follow from the smoothing-theory results (3) and (4). One also has to compute the Bogoliubov expectation values to the desired order. With the inhomogeneous Bogoliubov Hamiltonian (12) at hand, this is again a standard task in perturbation theory using the Nambu formalism [28].

Let us first check the homogeneous case $V = 0$. Then, (21) shrinks to $\delta n^{(0)} = \frac{1}{4} \int \frac{\mathrm{d}^d k}{(2\pi)^d} \left[a_k - a_k^{-1}\right]^2$, a well-known result [15]. In $d = 3$, this evaluates to a depletion density $\delta n^{(0)} = [6\sqrt{2}\pi^2\xi^3]^{-1}$ or equivalently to the relative depletion $\delta n^{(0)}/n = 8(na_s^3)^{1/2}/3\sqrt{\pi}$. In $d = 2$, one finds $\delta n^{(0)} = [8\pi\xi^2]^{-1}$. The $d = 1$ integral is infrared divergent, consistent with the fact that zero-point fluctuations prevent homogeneous 1D BECs. Cutting off the integral at some value $\alpha = \xi k_{\mathrm{IR}} \ll 1$, with k_{IR} of the order of the inverse system size, one finds $\delta n^{(0)} = (2\ln 2 - 2 - \ln\alpha)/(2\sqrt{2}\pi\xi)$, up to order α.

Now we evaluate the disorder-induced depletion by expanding all contributions to (21) to second order in V. Upon taking the ensemble average, terms of order V average to zero, and $\overline{\delta n} = \delta n^{(0)} + \overline{\delta n}^{(2)} + O(v^3)$. Each of the second-order terms is proportional to the correlator (15), and we can collect all contributions into a single kernel:

$$\overline{\delta n}^{(2)} = v^2 \delta n^{(0)} \int \frac{\mathrm{d}^d q}{(2\pi)^d} \sigma^d C_d(q\sigma) G_d(q\xi). \tag{22}$$

Details of this derivation will be given elsewhere [30]. Here we note that the relative depletion in units of v^2, $\Delta(\zeta) = \overline{\delta n}^{(2)}/v^2\delta n^{(0)}$ is only function of $\zeta = \sigma/\xi$. [31] This quantity is plotted in Fig. 2. As for the sound velocity correction, details depend on the specific correlation. In the δ-correlated limit $\zeta \to 0$, Fig. 2 shows the generic scaling $\Delta(\zeta) = \beta_d \zeta^d C_d(0)$ with numerical coefficients $\beta_1 \approx 0.235$ (weakly dependent on the cutoff α), $\beta_2 \approx 0.135$, $\beta_3 \approx 0.160$. In the limit $\zeta \to \infty$ of a very smooth potential, we find $\Delta \to G_d(0)$ with $G_3(0) = 3/8$, $G_2(0) = 0$, and $G_1(0) = -1/8$. In all cases but the last,

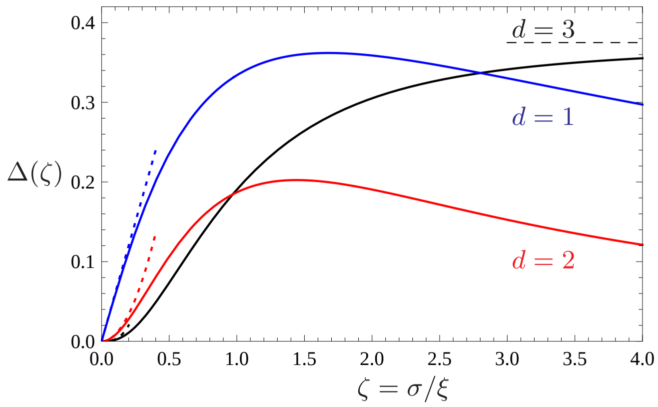

FIG. 2. Disorder-induced condensate depletion (22), relative to the clean value and in units of disorder strength v^2, as function of the correlation ratio $\zeta = \sigma/\xi$ for the Gaussian correlation (20). Dashed and dotted: universal limits as collected in Tab. II.

$\Delta(\zeta)$	$d=1$	$d=2$	$d=3$
$\zeta \gg 1$	$-\frac{1}{8}$	0	$\frac{3}{8}$
$\zeta \ll 1$	$\beta_1 C_1(0)\zeta$	$\beta_2 C_2(0)\zeta^2$	$\beta_3 C_3(0)\zeta^3$

TABLE II. Universal limits of the disorder-induced condensate depletion, relative to the clean value and in units of v^2 as plotted in Fig. 2, for very smooth disorder ($\zeta \gg 1$) and δ-correlated disorder ($\zeta \ll 1$). For the latter case, the numerical coefficients are $\beta_1 \approx 0.235$ (for $\alpha = 0.01$), $\beta_2 \approx 0.135$, $\beta_3 \approx 0.160$.

the condensate depletion due to disorder is positive, as expected. As shown in Fig. 2, also in $d = 1$ the depletion is positive for most values of ζ. The curve only crosses over to negative values for such a large value $\zeta = \sigma/\xi$ depending on the cutoff α, that the correlation length σ has to be comparable to the system size, which is not the regime of present interest.

In all cases, the combined depletion due to interaction and disorder reads $\overline{\delta n} = \delta n^{(0)}[1 + v^2\Delta(\zeta)]$, with $\Delta(\zeta)$ at most of the order of unity. Clearly, the fractional depletion induced by the disorder, $\overline{\delta n}^{(2)}/n = (\delta n^{(0)}/n)v^2\Delta(\zeta)$ is a factor $\delta n^{(0)}/n \ll 1$ smaller than the mean-field condensate deformation, which is of order v^2. In hindsight, this result is rather plausible: the primary effect of the external disorder potential is merely to deform the condensate. The depletion of the condensate itself is a secondary scattering effect, mediated by the weak repulsive boson interaction, and therefore considerably weaker.

In conclusion, we report substantial progress in the analytical description of interacting, condensed bosons in correlated disorder of any dimensionality. We derive the fundamental Bogoliubov Hamiltonian for excitations. This determines a wealth of (thermo-)dynamic quantities, out of which we calculate the sound velocity in all dimensions. Moreover, we calculate the disorder-induced quantum depletion, which proves to be much smaller than the previously known mean-field condensate deformation. We conclude that our theory should fare very well in describing the excitations of disordered interacting bosons, especially in dilute cold gases, where the study of well-controlled disorder in earnest has just begun [20–22].

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[31] Only in $d = 1$, it depends also weakly on the cutoff α that regularizes already the clean depletion; plots in this paper are done with $\alpha = 0.01$. We stress that our calculations require *no additional ad-hoc cutoffs*, neither infrared (since the excitations are orthogonal to the vacuum) nor ultraviolet (since potential correlations are included).